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REGULARITY OF LORENTZIAN BUSEMANN FUNCTIONS

GREGORY J. GALLOWAY AND ARNALDO HORTA

Abstract. A general theory of regularity for Lorentzian Busemann functions in future timelike geodesically complete spacetimes is presented. This treatment simplifies and extends the local regularity developed by Eschenburg, Galloway and Newman to prove the Lorentzian splitting theorem. Criteria for global regularity are obtained and used to improve results in the literature pertaining to a conjecture of Bartnik.

1. Introduction

Busemann functions were first introduced in Lorentzian geometry in the paper of Beem et al. [B+], and subsequently used by Eschenburg [E2], Galloway [G4] and Newman [N] to prove the Lorentzian version of the Cheeger-Gromoll splitting theorem as conjectured by S.-T. Yau [Y]. There are a number of technical difficulties one encounters when working with Lorentzian Busemann functions. For instance, Lorentzian Busemann functions are not in general continuous. In the proof of the Lorentzian splitting theorem, as developed in the aforementioned papers, various regularity properties related to the Lorentzian Busemann function are established locally, i.e., near the given timelike line. The proofs of some of these regularity results, especially in the nonglobally hyperbolic case considered by Newman, involved some fairly technical estimates.

In the present paper we develop a general theory of regularity of Lorentzian Busemann functions which simplifies and extends the previous treatments of regularity. The development we give is for spacetimes which are future timelike geodesically complete, but not necessarily globally hyperbolic. A similar development (with numerous simplifications) can be carried out for globally hyperbolic spacetimes. Our treatment is based on a generalization of the timelike co-ray condition introduced in [B+], but also utilizes key ingredients from [E2], [G4], [N] and [EG]. We mention, in particular, the notion of limit maximizing causal curves first introduced by Beem and Ehrlich ([BE1], [BE2]) and a key observation of Newman concerning the existence of maximal timelike geodesic segments in future timelike geodesically complete spacetimes. Our proofs rely primarily on "soft" methods, e.g., limit curve arguments and the reverse triangle inequality. In particular, all the technical estimates appearing in [E2] and [N] used to prove local regularity are eliminated. As a result, many aspects of the proof of the Lorentzian splitting theorem can now be substantially simplified. At the same time, we obtain new useful criteria for global regularity. We are able to use these global regularity results to improve results of

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Bartnik [B2] and Eschenburg and Galloway [EG] concerning a splitting conjecture for spatially closed spacetimes.

In Section 2 we present some basic background material. In Section 3 we establish criteria for the continuity of the Lorentzian Busemann function and for the smoothness of certain support functions. This entails, in particular, establishing a suitable existence result for maximal timelike geodesic segments. In Section 4 we consider the regularity of the level sets of the Lorentzian Busemann function. It is shown that, under appropriate circumstances, the level sets are acausal C^0 spacelike hypersurfaces which, when spacetime obeys a suitable energy condition, are mean convex (in the sense of support functions). In Section 5 we establish explicit regions of regularity and consider some applications.

2. Preliminaries

In this section we present a number of basic notions and results essential to our development. Most of the results presented here (or close forms of them) are known, and hence many proofs are omitted. For standard facts about spacetime and standard notations used but not defined below (such as $I^\pm$, $J^\pm$, $D^\pm$, $H^\pm$) we refer the reader to the standard references [BE2], [HE], [ON] and [P].

Limit curves.

Let (M, g) be a spacetime, i.e. a time-oriented Lorentzian manifold. We fix a complete Riemannian metric h on M. With the exception of certain limit curves which inherit a limit parameter, it will be convenient, in this section and the next, to parameterize all causal (i.e. nonspacelike) curves by arc length with respect to h. Hence, a causal curve γ is inextendible if and only if it is defined on $(-\infty, \infty)$ when parameterized with respect to h.

Extracting limit curves is a basic tool in causal theory. We have found the following version of the limit curve lemma to be particularly useful (cf. [EG], [G3]).

Lemma 2.1 (Limit Curve Lemma). *Let $\gamma_n : (-\infty, \infty) \to M$ be a sequence of causal curves (parameterized with respect to arc length in h). Suppose that $p \in M$ is an accumulation point of the sequence $\{\gamma_n(0)\}$. Then there exists an inextendible causal curve $\gamma : (-\infty, \infty) \to M$ such that $\gamma(0) = p$ and a subsequence $\{\gamma_m\}$ which converges to γ uniformly (with respect to h) on compact subsets of $\mathbb{R}$. γ is called a limit curve of $\{\gamma_n\}$.*

The proof of this lemma is an application of Arzela's theorem and is essentially contained in the proof of Proposition 2.18 in [BE2]. One advantage to this formulation is the fact that one can establish the upper semicontinuity of the Lorentzian arc length functional without invoking strong causality (cf. [EG],[G3]).

Proposition 2.2. *The Lorentzian arc length functional is upper semicontinuous with respect to the topology of uniform convergence on compact subsets, i.e., if a sequence $\gamma_n : [a, b] \to M$ of causal curves converges uniformly to the causal curve $\gamma : [a, b] \to M$, then*

$$L(\gamma) \geq \limsup_{n\to\infty} L(\gamma_n)\,. \tag{2-1}$$

The limit curve lemma was stated for inextendible causal curves. There is an obvious version of the limit curve lemma for future (respectively, past) inextendible causal curves, as well.

Let $d : M \times M \to [0, \infty]$ denote the Lorentzian distance function, i.e., if $q \in J^+(p)$ define

$$d(p, q) = \sup\{L(\gamma) : \gamma \in C(p, q)\} ,$$

where $C(p, q)$ is the set of all future directed causal curves from p to q, while if $q \notin J^+(p)$ define $d(p, q) = 0$. The Lorentzian distance function obeys the (reverse) triangle inequality, i.e., for $p \le q \le r$,

$$d(p, r) \ge d(p, q) + d(q, r) .$$

d is lower semicontinuous (see e.g. [BE2]) but, in the absence of global hyperbolicity, need not be continuous. This is a problem with which we shall have to contend.

The notion of limit maximizing causal curves introduced by Beem and Ehrlich [BE2] is especially useful for the construction of maximal causal geodesics in the absence of global hyperbolicity. A sequence $\gamma_n : [a_n, b_n] \to M$ of future directed causal curves is said to be *limit maximizing* if $L(\gamma_n) \ge d(\gamma_n(a_n), \gamma_n(b_n)) - \epsilon_n$ for some sequence $\epsilon_n \to 0$. Note that, if $p \le q$ and $d(p, q) < \infty$ then there exists a limit maximizing sequence of causal curves from p to q. One may use the upper semicontinuity of L and the lower semicontinuity of d to establish the following (cf. [BE2], [EG]).

Proposition 2.3. *Suppose that $\gamma_n : [a_n, b_n] \to M$ is a limit maximizing sequence of future directed causal curves that converges uniformly on some subinterval $[a, b] \subset \bigcap_n [a_n, b_n]$ to $\gamma : [a, b] \to M$. Then, $L(\gamma) = d(\gamma(a), \gamma(b))$, and hence γ is a future directed maximal geodesic from $\gamma(a)$ to $\gamma(b)$.*

Proof. Since the argument occurs frequently, we include the proof. Setting $\gamma_n = \gamma_n|_{[a,b]}$ we have

$$\begin{aligned}\limsup_{n\to\infty} L(\gamma_n) \le L(\gamma) \le d(\gamma(a), \gamma(b)) &\le \liminf_{n\to\infty} d(\gamma_n(a), \gamma_n(b)) \\ &\le \liminf_{n\to\infty} (L(\gamma_n) + \epsilon_n) = \liminf_{n\to\infty} L(\gamma_n) ,\end{aligned}$$

and thus,

$$L(\gamma) = \lim_{n\to\infty} L(\gamma_n) = \lim_{n\to\infty} d(\gamma_n(a), \gamma_n(b)) = d(\gamma(a), \gamma(b)) . \tag{2-2}$$

Rays and Co-rays.

A *ray* in M is a future inextendible causal geodesic $\gamma : [0, \infty) \to M$ each segment of which is maximal, $L(\gamma|_{[a,b]}) = d(\gamma(a), \gamma(b)), 0 \le a \le b$. We shall frequently construct rays as limits of certain limit maximizing curves. The following lemma is proved in [EG] by a fairly straightforward limit curve argument.

Lemma 2.4. *Let z_n be a sequence in M with $z_n \to z$. Let $p_n \in I^+(z_n)$ with $d(z_n, p_n) < \infty$. Let $\gamma_n : [0, a_n] \to M$ be a limit maximizing sequence of future directed causal curves with $\gamma_n(0) = z_n$ and $\gamma_n(a_n) = p_n$. Let $\tilde\gamma_n : [0, \infty) \to M$ be any future inextendible extension of γ_n. Suppose that $d(z_n, p_n) \to \infty$. Then any limit curve $\gamma : [0, \infty) \to M$ of the sequence $\tilde\gamma_n$ is a ray starting at z.*

Comment. The condition $d(z_n, p_n) \to \infty$ is used to show that $a_n \to \infty$. This condition does not imply that $\{p_n\}$ diverges to infinity (as it would were d a Riemannian distance function).

We extend the notion of a ray as follows. Let $S \subset M$ be a subset of M, and $\gamma : [0, \infty) \to M$ be a future inextendible causal curve. We say that γ is an S-ray if γ maximizes distance to S, i.e. if

$$L(\gamma|_{[0,a]}) = d(S, \gamma(a)) \quad \text{for all } a > 0\,,$$

where $d(S,q) = \sup\{d(p,q) : p \in S\}$. An S-ray γ is necessarily a ray emanating from $\gamma(0) \in S$. Conversely, a ray γ is a $\{\gamma(0)\}$-ray. In applications, the most important cases are this case and the case where S is a spacelike hypersurface. We observe that if γ is an S-ray then we have

$$d(p,q) < \infty \quad \text{for all } p, q \in I^-(\gamma) \cap I^+(S) \text{ with } p \leq q\,. \tag{2-3}$$

Indeed, by the triangle inequality we have that for r sufficiently large,

$$d(S,p) + d(p,q) + d(q,\gamma(r)) \leq d(\gamma(0),\gamma(r))\,. \tag{2-4}$$

Now suppose that $\gamma : [0,\infty) \to M$ is an S-ray and let $z \in I^-(\gamma) \cap I^+(S)$. Let $z_n \to z$ and put $p_n = \gamma(r_n)$ where $r_n \to \infty$. Then $z_n \in I^-(p_n)$ and $d(z_n, p_n) < \infty$ for all n sufficiently large. If γ is of infinite length then the triangle inequality implies that $d(z_n, p_n) \to \infty$. Therefore, if μ_n is a sequence of limit maximizing curves from z_n to p_n, then by Lemma 2.4, any limit curve $\mu : [0, \infty) \to M$ of the (suitably extended) μ_n's is a ray starting at z. A ray constructed in this fashion is called a *generalized co-ray* of γ. If the μ_n's are actually maximizers, $L(\mu_n) = d(z_n, p_n)$, then the limit is called a *co-ray* of γ. Finally, if $z_n = z$ for all n, we say that the co-ray μ is an *asymptote* of γ.

Co-rays in the globally hyperbolic setting were first introduced in [B+], while generalized co-rays were introduced in [EG].

The Lorentzian Busemann Function.

Let $\gamma : [0,\infty) \to M$ be a future complete timelike S-ray. The Busemann function $b : M \to [-\infty, \infty]$ associated to γ is defined as follows,

$$b(x) = \lim_{r\to\infty} b_r(x)\,,$$

where,

$$b_r(x) = d(\gamma(0),\gamma(r)) - d(x,\gamma(r))\,.$$

This limit always exists in the extended reals. If $x \in M \setminus I^-(\gamma)$ then $b_r(x) = d(\gamma(0),\gamma(r)) \to \infty$ as $r \to \infty$, and hence,

$$b(x) = \infty \quad \text{for all } x \in M \setminus I^-(\gamma)\,.$$

If $x \in I^-(\gamma)$, then b_r decreases monotonically in r, since for $s > r$ we have

$$d(x,\gamma(s)) \geq d(x,\gamma(r)) + d(\gamma(r),\gamma(s)), \quad \text{and}$$
$$d(\gamma(0),\gamma(s)) = d(\gamma(0),\gamma(r)) + d(\gamma(r),\gamma(s))\,.$$

Hence, the limit exists in this case also, and $b(x) < \infty$ for all $x \in I^-(\gamma)$. Since b_r is upper semicontinuous and, on $I^-(\gamma)$, b is the decreasing limit of the b_r's, b is upper semicontinuous on $I^-(\gamma)$.

Inequality (2-4) implies that if $x \in I^-(\gamma) \cap I^+(S)$ then for r sufficiently large,

$$d(S,x) + d(x,\gamma(r)) \le d(\gamma(0),\gamma(r))\,,$$

from which it follows that

$$b(x) \ge d(S,x) \ge 0 \text{ for all } x \in I^-(\gamma) \cap I^+(S). \tag{2-5}$$

Thus, both b and d are finite valued on $I^-(\gamma) \cap I^+(S)$.

Another simple application of the triangle inequality shows that

$$b(q) \ge b(p) + d(p,q) \text{ for all } p,q \in I^-(\gamma) \cap I^+(S) \text{ with } p \le q\,, \tag{2-6}$$

and, hence, b is nondecreasing along causal curves.

The next lemma establishes the existence of a useful class of support functions for b analogous to the support functions in the Riemannian case introduced in [EH].

Lemma 2.5. *Let $\gamma : [0,\infty) \to M$ be a future complete S-ray in a spacetime M, and let b be the associated Busemann function. If $\alpha : [0,\infty) \to M$ is a timelike asymptote to γ starting at $p \in I^-(\gamma) \cap I^+(S)$, then for each $s > 0$,*

$$b_{p,s}(x) = b(p) + d(p,\alpha(s)) - d(x,\alpha(s)) \tag{2-7}$$

is an upper support function for b at p, i.e., $b_{p,s}(x) \ge b(x)$ for all x near p with equality holding when $x = p$.

Proof. Suppose that $\alpha_n : [0,a_n] \to M$ is the sequence of maximal geodesic segments that converge to α. We have $\alpha_n(0) = p$ and $\alpha_n(a_n) = \gamma(r_n)$, where $r_n \to \infty$ (by construction) and $a_n \to \infty$ (by the remark after Lemma 2.4). Then, by the triangle inequality and the lower semicontinuity of d, we have that, for fixed $s > 0$, $x \in I^-(\alpha(s)) \cap I^+(S)$ and n sufficiently large,

$$\begin{aligned} b_{r_n}(x) - b_{r_n}(p) &= d(p,\gamma(r_n)) - d(x,\gamma(r_n)) \\ &\le d(p,\gamma(r_n)) - d(x,\alpha_n(s)) - d(\alpha_n(s),\gamma(r_n)) \\ &= d(p,\alpha_n(s)) - d(x,\alpha_n(s)) \\ &\le d(p,\alpha_n(s)) - d(x,\alpha(s)) + \delta_n \end{aligned} \tag{2-8}$$

where $\delta_n \to 0$. Note that (2-2) implies $d(p,\alpha_n(s)) \to d(p,\alpha(s))$ as $n \to \infty$. Thus, letting $n \to \infty$ in (2-8) we obtain

$$b(x) \le b(p) + d(p,\alpha(s)) - d(x,\alpha(s)) \tag{2-9}$$

as desired.

The inequality that results by setting $x = \alpha(t)$, $t < s$ in (2-9), together with (2-6), yields the following analogue of the well-known Riemannian result that the Busemann function increases with unit speed along asymptotes.

Proposition 2.6. *Let $\gamma : [0,\infty) \to M$ be a future complete S-ray in a spacetime M, and let b be the associated Busemann function. If $\alpha : [0,\infty) \to M$ is a timelike asymptote to γ starting at $p \in I^-(\gamma) \cap I^+(S)$, then*

$$b(\alpha(t)) = d(p, \alpha(t)) + b(p)$$

for all $t > 0$.

The last two results were obtained by Eschenburg [E2] in the globally hyperbolic setting.

Comment. We have seen that if γ is an S-ray, then the associated Busemann function and the Lorentzian distance function are finite valued on $I^-(\gamma) \cap I^+(S)$. For this reason, in our presentation of the results in the next two sections, we restrict attention to this region. However, in the special case that S is an acausal spacelike hypersurface, it can be shown that both b and d are finite valued on the larger region $I^-(\gamma) \cap I^+(D^-(S)) = I^-(\gamma) \cap [D^-(S) \cup J^+(S)]$. The results described at the end of this section and in the next two sections extend in a straightforward manner to this larger region. We return to this point in Section 5.

3. Continuity of the Lorentzian Busemann Function

In this section we establish criteria for the continuity of Busemann functions in spacetimes which are assumed to be future timelike geodesically complete. We also establish criteria for the existence of timelike asymptotes and the smoothness of the support functions $b_{p,s}$ introduced in Section 2. The entire development hinges on a generalization of the timelike co-ray condition introduced in [B+] in the globally hyperbolic setting.

One of the main obstacles to proving continuity, which we now address, is the lack of a suitable existence result for maximal geodesics. Until further notice we continue to parameterize all causal curves with respect to the given background Riemannian metric h.

Existence of maximizers.

In the following proposition we abstract a key observation of Newman ([N], Lemma 3.9).

Proposition 3.1. *Let M be a future timelike geodesically complete spacetime. Suppose p and q are points in M with $q \in I^+(p)$ and $d(p,q) < \infty$. Let $\gamma_n : [0, a_n] \to M$ be a limit maximizing sequence of future directed causal curves from p to q. For each n, let $\tilde{\gamma}_n : [0,\infty) \to M$ be a future inextendible extension of γ_n. Then, if $\gamma : [0,\infty) \to M$ is a limit curve of $\{\tilde{\gamma}_n\}$ either γ maximizes from p to q or γ is a null ray.*

Proof. Let $a = \sup_n a_n$; note that $a > 0$. By passing to a subsequence, we may assume $a_n \to a$. Then Proposition 2.3 implies that $\gamma|_{[0,a)}$ is a maximal geodesic. There are two cases to consider.

First, suppose $a < \infty$. We know that $\gamma|_{[0,a)}$ extends continuously to a, hence, $\gamma|_{[0,a)}$ extends to a as a geodesic which is easily seen to be maximal. Note that $\gamma(0) = p$ and $\gamma(a) = \lim_{n\to\infty} \gamma_n(a_n) = q$. Hence, $\gamma|_{[0,a]}$ is the maximizer sought.

Now suppose that $a = \infty$. If γ is null, then γ is a null ray, and we are done. If γ is timelike then, by the completeness assumption, γ has infinite length. This together with the finiteness of the distance function implies that there exists $l > 0$ such that, $L(\gamma|_{[0,l]}) > d(p,q)$. Since $a_n \to \infty$, (2-2) implies, $L(\gamma_n|_{[0,l]}) \to L(\gamma|_{[0,l]})$. Hence, for n sufficiently large,

$$L(\gamma_n|_{[0,a_n]}) > L(\gamma_n|_{[0,l]}) > d(p,q)\,.$$

This is a contradiction, since $\gamma_n : [0,a_n] \to M$ is a causal curve segment from p to q and $d(p,q)$ is the supremum of the lengths of all such curves. Hence γ can not be timelike.

We now introduce the condition that is crucial to our entire treatment of regularity.

Definition 3.2. Let M be a future timelike geodesically complete spacetime and let γ be a timelike S-ray. Suppose that $p \in I^-(\gamma) \cap I^+(S)$. We say that the *generalized timelike co-ray condition* holds at p if every generalized co-ray to γ starting at p is timelike.

This condition is a natural generalization of the timelike co-ray condition introduced in [B+]. As the next result shows, this condition enables us to establish the existence of maximal timelike geodesic segments between certain points of spacetime.

Lemma 3.3. *Let M be a future timelike geodesically complete spacetime and let γ be a timelike S-ray. Assume the generalized timelike co-ray condition holds at $p \in I^-(\gamma) \cap I^+(S)$. Then there exists a neighborhood U of p and a positive number R such that for all $q \in U$ and for all $r > R$, there exists a maximal timelike geodesic segment from q to $\gamma(r)$.*

Proof. Suppose not. Then there exist sequences $r_n \to \infty$ and $p_n \to p$, with $\gamma(r_n) \in I^+(p_n)$, such that there is no maximal timelike geodesic from p_n to $\gamma(r_n)$. We can assume $p_n \in I^-(\gamma) \cap I^+(S)$, and hence, $d(p_n, \gamma(r_n)) < \infty$ for all n. Then, Proposition 3.1 implies that for each n there exists a sequence $\alpha_{nk} : [0, a_{nk}] \to M$ of limit maximizing curves,

$$L(\alpha_{nk}) \ge d(\alpha_{nk}(0), \alpha_{nk}(a_{nk})) - \epsilon_{nk}, \quad \lim_{k\to\infty} \epsilon_{nk} = 0\,,$$

from $\alpha_{nk}(0) = p_n$ to $\alpha_{nk}(a_{nk}) = \gamma(r_n)$, with $\lim_{k\to\infty} a_{nk} = \infty$, which converges to a null ray $\alpha_n : [0,\infty) \to M$ starting at p_n. Equation (2-2) implies that for any $s > 0$, $L(\alpha_{nk}|_{[0,s]}) \to L(\alpha_n|_{[0,s]}) = 0$ as $k \to \infty$.

Thus we can choose an increasing sequence $k_n \to \infty$ such that the following conditions hold, where $\eta_n = \alpha_{nk_n}$ and $b_n = a_{nk_n}$.

1. $b_n \to \infty$.
2. $L(\eta_n|_{[0,1]}) < \frac{1}{n}$ for all n.
3. $L(\eta_n|_{[0,b_n]}) \ge d(\eta_n(0), \eta_n(b_n)) - \frac{1}{n}$, for all n.

Conditions (1) and (3), together with Proposition (2.3), imply that the η_n's converge to a ray η based at p. Condition (2) implies that η is null. Hence, η is a null generalized co-ray at p, contrary to our assumptions.

Comment. A similar argument shows that the generalized timelike co-ray condition is an open condition, i.e., if the the generalized timelike co-ray condition holds at $p \in I^-(\gamma) \cap I^+(S)$, then it holds in a neighborhood of p.

The next result shows that the maximal timelike geodesic segments guaranteed by Lemma 3.3 obey a certain uniformity condition, namely, their initial tangents are locally bounded away from the light cones. It is convenient at this point to introduce an exhaustion by compact sets of UT_pM, the set of unit timelike vectors at p. For each $p \in M$ and each $C > 0$, let $K_C(p)$ denote the set of all vectors $X \in UT_pM$ such that $h(X, X) \le C$, where h is the given background Riemannian metric. The sets $K_C(p)$, $C > 0$, are compact and nonempty for C sufficiently large, they increase as C increases, and they exhaust UT_pM, $\cup_{C>0} K_C(p) = UT_pM$.

Lemma 3.4. *Let M be a future timelike geodesically complete spacetime and let γ be a timelike S-ray. Assume the generalized timelike co-ray condition holds at $p \in I^-(\gamma) \cap I^+(S)$. Then there exists a neighborhood U of p and positive constants R and C such that for all $q \in U$ and $r > R$, if $\alpha : [0, a] \to M$ is any maximal timelike geodesic segment from q to $\gamma(r)$, parameterized with respect arc length in the Lorentzian metric, then $\alpha'(0) \in K_C(q)$.*

Comment. When dealing with timelike geodesic segments, there are two notions of convergence, as limit curves and by accumulation of the initial directions. The main point of the proof that follows is that, in the appropriate context, these two notions coincide.

Proof. Suppose to the contrary, the lemma is false. Then there are sequences $p_n \to p$, $r_n \to \infty$, and maximal timelike geodesic segments $\alpha_n : [0, a_n] \to M$ from p_n to $\gamma(r_n)$ parameterized with respect to Lorentzian arc length such that $h(\alpha_n'(0), \alpha_n'(0)) \to \infty$ as $n \to \infty$. Let $\bar{\alpha}_n$ be the reparameterization of α_n with repect to h. By passing to a subsequence, we can assume that the $\bar{\alpha}_n$'s converge (in the sense of Lemma 2.4) to a ray $\bar{\alpha}$ starting at p. Let α be its reparameterization with respect to Lorentzian arc length.

Since the generalized timelike co-ray condition holds at p, $\bar{\alpha}$ must be timelike. Choose $\delta > 0$ so that $\bar{\alpha}([0, \delta])$ is contained in a convex neighborhood of p. Let $\epsilon_n = d(\bar{\alpha}_n(0), \bar{\alpha}_n(\delta))$ and $\epsilon = d(\bar{\alpha}(0), \bar{\alpha}(\delta))$; note that $\epsilon > 0$. By (2-2), $\epsilon_n \to \epsilon$. Furthermore, by properties of the exponential map, $\epsilon_n \alpha_n'(0) = exp_{p_n}^{-1}(\bar{\alpha}_n(\delta)) \to exp_p^{-1}(\bar{\alpha}(\delta)) = \epsilon\alpha'(0)$. Thus, $\alpha_n'(0) \to \alpha'(0)$ and hence, $h(\alpha_n'(0), \alpha_n'(0)) \to h(\alpha'(0), \alpha'(0))$, which is a contradiction.

As an immediate corollary to Lemma 3.4 we have the following.

Corollary 3.5. *Let M be a future timelike geodesically complete spacetime and let γ be a timelike S-ray. Assume the generalized timelike co-ray condition holds at $p \in I^-(\gamma) \cap I^+(S)$. Then there exists a neighborhood U of p and a positive constant C such that for all $q \in U$, if $\alpha : [0, \infty) \to M$ is any timelike asymptote (or co-ray) from q, parameterized with respect to arc length in the Lorentzian metric, then $\alpha'(0) \in K_C(q)$.*

Comment. Let $p \in I^-(\gamma) \cap I^+(S)$ be a point at which the generalized timelike co-ray condition holds. A neighborhood U of p in $I^-(\gamma) \cap I^+(S)$ for which Lemmas 3.3 and 3.4 and Corollary 3.5 hold shall be referred to as a *nice* neigborhood of p.

Continuity of the Lorentzian Busemann function.

We give a proof of continuity of the Lorentzian Busemann function along the lines of Eschenburg [E2]. For this we need to establish some partial continuity of the Lorentzian distance function.

Proposition 3.6. *Let M be a future timelike geodesically complete spacetime and let γ be a timelike S-ray. Assume the generalized timelike co-ray condition holds at $p \in I^-(\gamma) \cap I^+(S)$. Then there exists a neighborhood U of p and a positive number R such that for all $r > R$, the function $\delta : U \to [0, \infty)$ defined by $\delta(x) = d(x, \gamma(r))$ is continuous on U.*

Proof. Let U, R and C be given as in Lemma 3.4. Moreover, by Lemma 3.3, we may assume that for all $x \in U$ and $r > R$ there exists a maximal timelike geodesic segment from x to $\gamma(r)$. Fix $r > R$ and set $q = \gamma(r)$. It suffices to show that δ is upper semicontinuous on U.

If δ is not upper semicontinuous at $x \in U$ then there exists a sequence $x_n \to x$ such that $\lim_{n\to\infty} d(x_n, q) > d(x, q)$. For each n, let α_n be a maximal timelike geodesic segment, parameterized with respect to Lorentzian arc length, from x_n to q. The initial tangents $\{\alpha_n'(0)\}$ are contained in a compact subset of the unit timelike tangent bundle. By this fact and the assumption of future completeness, there exists a subsequence $\{\alpha_{n_k}\}$ that converges to a maximal timelike geodesic segment α from x to $\gamma(r)$. This convergence implies, $d(x_{n_k}, q) = L(\alpha_{n_k}) \to L(\alpha) = d(x, q)$, which is a contradiction.

Theorem 3.7. *Let M be a future timelike geodesically complete spacetime and let γ be a timelike S-ray. Assume the generalized timelike co-ray condition holds at $p \in I^-(\gamma) \cap I^+(S)$. Then the Busemann function b associated to γ is Lipschitz continuous on a neighborhood of p.*

The proof is an application of the following simple lemma, which is proved in the appendix of [E2].

Lemma 3.8. *Let U be an open convex domain in $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ a continuous function. Assume that for any $q \in U$ there is a smooth lower support function f_q at q such that $\|d(f_q)_q\| \le L$. Then f is Lipschitz with Lipschitz constant L.*

Proof of Theorem 3.7. The proof in [E2] can now be implemented with only minor modifications. Choose U, R and C as in Lemmas 3.3 and 3.4. Let $(\mathcal{N}, \phi)$ be a convex normal coordinate neighborhood of p. By choosing U sufficiently small we may assume that the closure of U is compact and contained in $\mathcal{N}$, and that $\phi(U)$ is convex in $\mathbb{R}^n$ ($n = \dim M$). Let h_0 be the Euclidean metric, $h_0 = \sum_{i=1}^n dx_i^2$, where the x_i's are the local coordinates of $(\mathcal{N}, \phi)$. There is a constant K such that $h_0 \le Kh$ on U, where h is the given background Riemannian metric on M.

We show that the functions, $b_r = r - d_r$, $r > R$, where $d_r(x) = d(x, \gamma(r))$, are equi-Lipschitz continuous on U. By Proposition 3.6, we can assume that the functions d_r, $r > R$, are continuous on U.

Let α be a maximal timelike geodesic segment from a point $q \in U$ to $\gamma(r)$, and let $q' \neq q$ be a point on α in U. Consider the function $f_{q,r}$ defined by

$$f_{q,r}(x) = d_{\mathcal{N}}(x, q') + d(q', \gamma(r)),$$

where $d_{\mathcal{N}}$ is the local Lorentzian distance function on $\mathcal{N}$ (cf. [BE2]),

$$d_{\mathcal{N}}(x, y) = [-\langle \exp_x^{-1} y, \exp_x^{-1} y\rangle]^{\frac{1}{2}}, \quad y \in J^+(x, \mathcal{N}).$$

Near q, $x \to d_{\mathcal{N}}(x, q')$ is smooth and satisifies $d_{\mathcal{N}}(x, q') \le d(x, q')$, with equality holding at $x = q$. This fact and the triangle inequality imply that $f_{q,r}$ is a smooth lower support function of d_r at q. Moreover, for any $v \in T_qM$, we have the simple estimate

$$|df_{q,r}(v)| = |\langle \alpha'(0), v\rangle| \leq G|\alpha'(0)|_0\,|v|_0 \leq GKC^{\frac{1}{2}}|v|_0,$$

where $G = \sup\{|g_{ij}(x)| : x \in U, 1 \leq i, j \leq n\}$ and $|v|_0 = h_0(v,v)^{\frac{1}{2}}$. Thus, by Lemma 3.8, b_r, $r > R$, and hence b are Lipschitz continuous on U with Lipschitz constant $L = GKC^{\frac{1}{2}}$.

The cut locus and smoothness of the distance function.

In order to show that the support function $b_{p,s}$ defined in equation (2-7) is smooth near p, we need some control over the timelike cut locus. Unfortunately, in the absence of global hyperbolicity, the timelike cut locus need not be well-behaved (for example, being a timelike cut point to a given point need not be a symmetric condition). The following proposition concerning the timelike cut locus is sufficient for our purposes.

Proposition 3.9. *Let $\alpha : [0, r_0] \to M$ be a maximal timelike geodesic segment in a spacetime M, and assume for each $r \in (0, r_0)$, $x \to d(x, \alpha(r))$ is finite valued on a neighborhood of $\alpha(0)$. Then, for each $r \in (0, r_0)$, there exists a neighborhood U of the segment $\alpha([0, r])$ which does not meet the past timelike cut locus of $\alpha(r)$.*

Proof. It is sufficient to show that for each $s \in [0, r]$, there exists a neighborhood U of $\alpha(s)$ which does not meet the past timelike cut locus of $\alpha(r)$. There are three cases: $s = 0$, $0 < s < r$ and $s = r$.

Case (1) $s = r$. Since α is a maximal timelike geodesic it is easy to see that strong causality holds at $\alpha(r)$ (use a "cutting the corner" argument together with Lemma 4.16 in [P]). Hence, by Theorem 3.27 in [BE2], there exists a convex normal neighborhood U of $\alpha(r)$ such that the Lorentzian distance function agrees with the local Lorentzian distance function on U, $d = d_U$. It follows immediately that there are no past timelike cut points to $\alpha(r)$ in U.

Case (2) $0 < s < r$. This case is similar to the case $s = 0$, but simpler. We omit the details.

Case (3) $s = 0$. We must show that there is a neighborhood U of p that does not meet the past timelike cut locus of $\alpha(r)$. This is the only case that uses the finiteness assumption. The proof makes use of the following lemma.

Lemma 3.10. *Let the assumptions be as in Proposition* 3.9*. Then for each $r \in (0, r_0)$ there exists a neighborhood U of p such that for each $q \in U$, there exists a maximal timelike geodesic segment from q to $\alpha(r)$.*

Proof of Lemma 3.10. Suppose there exists a number $r \in (0, r_0)$ for which the conclusion is false. Then, in view of the finiteness of the distance function, there exists a sequence $p_n \to p$ and, for each n, a limit maximizing sequence of past directed causal curves $\sigma_{n,k} : [0, a_{n,k}] \to M$ such that, for each k, $\sigma_{n,k}(0) = \alpha(r)$ and $\sigma_{n,k}(a_{n,k}) = p_n$. Furthermore we may assume, by selecting a suitable subsequence, that as $k \to \infty$, $\sigma_{n,k} \to \sigma_n$ where σ_n is either a past inextendible null ray or a past inextendible timelike ray. (Recall the proof of Proposition 3.1 and note that we do not assume past completeness.) As was done in Lemma 3.3, we can diagonalize and obtain a limit maximizing sequence $\eta_n : [0, a_n] \to M$ (with $\eta_n(0) = \alpha(r)$, $\eta_n(a_n) = p_n$, and $a_n \to \infty$) which converges to a past inextendible ray $\eta : [0, \infty) \to M$ starting at $\alpha(r)$.

Fix $r_1 \in (r, r_0)$ and consider the curve $\hat{\eta}_n : [0, c_n] \to M$ obtained by concatenating each η_n with the segment along $-\alpha$ from r_1 to r and reparameterizing appropriately ($c_n = \ell + a_n$, where $\ell = r_1 - r$). We claim that this new sequence is limit maximizing.

To prove the claim, first note that $\hat{\eta}_n(0) = \alpha(r_1)$, $\hat{\eta}_n(\ell) = \alpha(r)$ and $\hat{\eta}_n(c_n) = p_n$. Then by the lower semicontinuity of the distance function and the fact that the original η_n's are limit maximizing, we have

$$\begin{aligned} L(\hat{\eta}_n|_{[0,c_n]}) &= d(\alpha(r), \alpha(r_1)) + L(\hat{\eta}_n|_{[\ell,c_n]}) \\ &\geq d(\alpha(r), \alpha(r_1)) + d(p_n, \alpha(r)) - \epsilon_n \\ &\geq d(\alpha(r), \alpha(r_1)) + d(p, \alpha(r)) - \epsilon_n - \delta_n \\ &= d(p, \alpha(r_1)) - \epsilon_n - \delta_n \,, \end{aligned}$$

where $\epsilon_n, \delta_n \to 0$. This implies the limit maximality of the sequence $\{\hat{\eta}_n\}$.

Since $\{\hat{\eta}_n\}$ is limit maximizing and each $\hat{\eta}_n$ contains the segment of α from $\alpha(r)$ to $\alpha(r_1)$, $\{\hat{\eta}_n\}$ converges to a past inextendible timelike ray $\hat{\eta}$ containing this same segment. Hence, $\hat{\eta}$ passes through p and yet, by construction, is contained in $\overline{J^+(p)}$. This implies that $p \in I^+(p)$, which contradicts the finiteness of the d.

We now complete the proof of case (3) of Proposition 3.9. In what follows we assume that all timelike geodesics (including α) are parameterized with respect to Lorentzian arc length. Suppose the proposition fails in this case. Then, there exist $r \in (0, r_0)$ and $p_n \to p$ such that p_n is a cut point of $\alpha(r)$ for all n. Since p and $\alpha(r)$ are not conjugate along α, there is a neighborhood V of $-\alpha'(r)$ in $T_{\alpha(r)}M$ which is diffeomorphic (via the exponential map) to a neighborhood U of p. By shrinking U if necessary, we can use Lemma 3.10 and assume that there are maximal timelike segments from $\alpha(r)$ to each point in U. Without loss of generality, assume that $p_n \in U$ for all n.

Let η_n be a maximal segment from $\alpha(r)$ to p_n. Now suppose η_n comes from a vector in V (i.e., $\eta_n'(0) \in V$). Then, since each p_n is a cut point, by following η_n further into the past, we obtain a point $q_n \in U$ such that η_n fails to maximize to q_n. But, by Lemma 3.10 there is some maximal segment from $\alpha(r)$ to q_n. Hence, since V is diffeomorphic to U via the exponential map, it follows that there is a vector v_n outside of V which gives the required maximal segment when exponentiated. By this reasoning, we can assume that for each n, there is a maximal segment from $\alpha(r)$ to p_n which has initial tangent outside of V. Then, by passing to a subsequence and reasoning as in Lemma 3.10, we obtain a maximal segment from $\alpha(r)$ to p whose initial tangent lies outside V. Since $-\alpha'(r) \in V$, this implies that there are two maximal segments from $\alpha(r)$ to p, which contradicts the fact that α is a ray. This completes the proof of the proposition.

Comment. Case (2) of Proposition 3.9 essentially corresponds to Lemma 3.10 in [N]. Newman avoids case (3) in his proof of the Lorentzian splitting theorem by the use of a certain trick.

It now follows from standard arguments (cf. [E2]) that, in the setting of Proposition 3.9, the distance function $x \to d_r(x) = d(x, \alpha(r))$ is smooth near p, with smooth future pointing unit timelike gradient, $\langle \nabla d_r, \nabla d_r \rangle = -1$. Moreover, if M obeys the *strong energy condition* (or, more correctly, the *timelike convergence condition*), $Ric(X, X) \geq 0$, for all timelike vectors X, then the usual Riccati equation argument gives

$$\triangle d_r(p) \geq -\frac{n-1}{d(p,\alpha(r))}\,, \tag{3-1}$$

where $n = \dim M$. In the next section this estimate will be applied to the support functions $b_{p,s}$ introduced in Lemma 2.5.

4. Regularity of the Level Sets

We now consider some basic regularity properties of the level sets of Busemann functions. Throughout this section, unless otherwise stated, timelike geodesics are parameterized with respect to arc length in the given Lorentzian metric $\langle , \rangle$.

Spacelike hypersurfaces.

We adopt the C^0 definition of spacelike hypersurfaces used in [EG].

Definition 4.1. A subset $S \subset M$ is called a spacelike hypersurface if for each $p \in S$, there is a neighborhood U of p in M such that $S \cap U$ is acausal and edgeless in U.

Comment. A spacelike hypersurface is necessarily an embedded topological submanifold of M of codimension one. A *smooth spacelike hypersurface*, by which we mean a smooth codimension one submanifold with everywhere timelike normal, is a spacelike hypersurface in the sense of our definition. A subset $S \subset M$ is called a partial Cauchy surface if it is acausal and edgeless in M. In particular, a partial Cauchy surface is a spacelike hypersurface and is closed as a subset of M.

Proposition 4.2. *Let M be a future timelike geodesically complete spacetime and let γ be a timelike S-ray. Suppose $p \in I^-(\gamma) \cap I^+(S)$ is a point on a level set $\{b = c\}$ where the generalized timelike co-ray condition holds. Then there exists a neighborhood U of p such that the set $\Sigma_c = U \cap \{b = c\}$ is acausal in M and edgeless in U. In particular, Σ_c is a partial Cauchy surface in U.*

Proof. A proof can be given along the lines of the proof of Lemma 2.3 in [G4]. Let U be a nice neighborhood of p (cf. the comment after Corollary 3.5). The inequality (2-6) shows that b is strictly increasing along future directed timelike curves (and hence, Σ_c is at least achronal). This fact and the continuity of b on U implies that Σ_c is edgeless in U.

If Σ_c is not acausal, there exists a pair of points $x, y \in \Sigma_c$ and a future directed null geodesic η joining them. Since y is in a nice neighborhood, there exists a sequence of maximal segments α_n from y to $\gamma(r_n)$ ($r_n \to \infty$) which converges to a timelike geodesic ray at y. By cutting the corner of the broken geodesic $\eta \cup \alpha_n$, and comparing with the corner of $\eta \cup \alpha$, we see there exists $\epsilon > 0$ such that for all n sufficiently large,

$$b_{r_n}(y) - b_{r_n}(x) = d(x, \gamma(r_n)) - d(y, \gamma(r_n)) > \epsilon\,.$$

This contradicts $b(x) = b(y)$.

Comment. The proof shows that Proposition 4.2 applies to any open set $U \subset I^-(\gamma) \cap I^+(S)$ on which the generalized timelike co-ray condition holds.

Convexity properties of the level sets and the energy condition.

In [G4] certain convexity properties of the level sets of Lorentzian Busemann functions in spacetimes which obey the strong energy condition were established. We now indicate how to extend these results to the present setting. In fact, since we are assuming future completeness, the proofs simplify in certain respects.

We begin by establishing a Hessian estimate for certain distance functions. This result replaces the technical lemma in [G4].

Lemma 4.3. *Let M be a future timelike geodesically complete spacetime and let γ be a timelike S-ray. Assume the generalized timelike co-ray condition holds at $p \in I^-(\gamma) \cap I^+(S)$. Then there exists a neighborhood U of p and constants $t_0 > 0$ and $A > 0$ such that for each $q \in U$ and for each timelike asymptote α from q we have*

$$\operatorname{Hess} d_t(v,v) \geq -A\langle v^\perp, v^\perp\rangle \tag{4-1}$$

for all $v \in T_qM$ and $t \geq t_0$, where $d_t = d(.,\alpha(t))$ and $v^\perp$ is the projection onto the normal space of $\alpha'(0)$.

Comment. Proposition 3.9 implies that for each $q \in U$ and each $t > 0$, d_t is smooth in a neighborhood of q.

Proof. Let U be a nice neighborhood of p with compact closure. By Corollary 3.5, the set of all initial tangents to asymptotes starting from points in U is contained in a certain compact subset of the unit timelike tangent bundle. Then, by taking U and t_0 to be sufficiently small, we can insure that the set of all tangents to the initial segments of length t_0 of asymptotes emanating from points in U is contained in a compact subset of the unit timelike bundle. Hence the set of all timelike planes Π containing these initial tangents is contained in a compact subset of the set off all timelike planes. Thus, the set of all sectional curvatures of such planes is bounded above by some constant k, $K(\Pi) \leq k$.

Now consider $B = -\operatorname{Hess}(d_{t_0})$ along the segment $\alpha|_{[0,t_0]}$. $B = B_s$ corresponds to the second fundamental form of the level set $d_{t_0} = s$, $0 \leq s < t_0$, and $u \to B_u$ ($u = t_0 - s$) obeys the so-called Riccati equation (Eq. (3) in [E1]). By applying the appropriate comparison theorem (e.g., Proposition 2.4 in [E1]), it follows immediately that there exists a constant $A = A(k)$ such that

$$\operatorname{Hess} d_{t_o}(v,v) \geq -A\langle v^\perp, v^\perp\rangle\,,$$

for all $v \in T_qM$. The lemma now follows from the well-known fact (cf., [E2]) that $t \to \operatorname{Hess} d_t(w,w)$ is an increasing function of t.

The following *maximum principle* generalizes in various respects Lemma 2.4 in [G4] and Lemma 3.14 in [N].

Proposition 4.4. *Let M be a future timelike geodesically complete spacetime which obeys the strong energy condition,* $\operatorname{Ric}(X,X) \geq 0$ *for all timelike vectors X, and let γ be a timelike S-ray. Let $W \subset I^-(\gamma) \cap I^+(S)$ be an open set on which the generalized timelike co-ray condition holds. Let $\Sigma \subset W$ be a connected smooth spacelike hypersurface with nonpositive mean curvature, $H_\Sigma \leq 0$. If the Busemann function $b = b_\gamma$ attains a minimum along Σ then b is constant along Σ.*

Comment. We are using here a sign convention opposite of that used in [G4]: $H_\Sigma = \operatorname{div}_\Sigma N$, where N is the future pointing unit normal along Σ.

Proof. The proof is similar to the proof of Lemma 2.4 in [G4]. Because of the assumption of future completeness the proof simplifies somewhat: one does not have to pass to the pre-Busemann functions b_r.

Suppose that b achieves a minimum $b(q) = a$ at $q \in \Sigma$. Let U be a nice neighborhood of q for which Lemma 4.3 also holds. Since $b|_\Sigma$ is continuous, it is sufficient to show that $b = a$ in a neighborhood of q in Σ. If it doesn't, there exists a coordinate ball B, $\bar{B} \subset \Sigma \cap U$, centered at q such that $\partial B \neq \partial^0 B$ where

$$\partial^0 B = \{x \in \partial B : b(x) = a\}\,.$$

It follows from Lemma 4.3 that there exists a constant $C > 0$ such that

$$\operatorname{Hess} d_t(v, v) \geq -C\,, \tag{4-2}$$

for all asymptotes at x, for all $x \in B$, for all $v \in T_x\Sigma$ with $\langle v, v\rangle \leq 1$, and for all t sufficiently large.

Note that $b > a$ on $\partial B \setminus \partial^0 B$. By choosing B sufficiently small, we can construct a smooth function h on Σ having the following properties (cf. [EH]).

1. $h(q) = 0$,
2. $|\nabla_\Sigma h| \leq 1$ on B, where ∇_Σ is the gradient operator on Σ,
3. $\Delta_\Sigma h \leq -D$ on B where D is a positive constant and Δ_Σ is the induced Laplacian on Σ, and
4. $h > 0$ on $\partial^0 B$.

Consider the function $f_\epsilon = b + \epsilon h$. Observe that $f_\epsilon(q) = a$ and, for ϵ sufficiently small, $f_\epsilon > a$ on ∂B. Thus, f_ϵ attains a minimum on B, say at p.

Let $\alpha : [0, \infty) \to M$ be a timelike asymptote to γ at p. Then we know that for each $t > 0$,

$$b_{p,t}(x) = b(p) + t - d(x, \alpha(t)) \tag{4-3}$$

is a smooth upper support function for b at p (cf. Lemma 2.5 and Propostion 3.9). Then the function $f_{\epsilon,t} = b_{p,t} + \epsilon h$ is a smooth upper support function for f_ϵ at p, which implies that $f_{\epsilon,t}$ also has a minimum at p. We will get a contradiction by calculating $\Delta_\Sigma f_{\epsilon,t}(p)$ and showing that it is negative for ϵ sufficiently small and t sufficiently large. To begin, we have

$$\Delta_\Sigma f_{\epsilon,t}(p) = \Delta_\Sigma b_{p,t}(p) + \epsilon \Delta_\Sigma h(p)\,. \tag{4-4}$$

The formula relating the spacetime Laplacian (d'Alembertian) Δ to Δ_Σ gives

$$\Delta_\Sigma b_{p,t} = \Delta b_{p,t} + H_\Sigma \langle \nabla b_{p,t}, N\rangle + \operatorname{Hess} b_{p,t}(N, N)\,, \tag{4-5}$$

where N is the future directed normal to Σ. Now, by (3-1) and (4-3) we have

$$\Delta b_{p,t}(p) \leq \frac{n-1}{t}\,. \tag{4-6}$$

Now, note that $\nabla_\Sigma f_{\epsilon,t}(p) = 0$ and $\nabla b_{p,t}(p) = -\alpha'(0)$. Using these, we get that $N = \langle N, \alpha'(0)\rangle^{-1}(-\alpha'(0) + \epsilon \nabla_\Sigma h)$. Using this expression for N, we obtain

$$\begin{aligned} \operatorname{Hess} b_{p,t}(N, N)|_p &= \epsilon^2 \langle N, \alpha'(0)\rangle^{-2} \operatorname{Hess} b_{p,t}(\nabla_\Sigma h, \nabla_\Sigma h)|_p \\ &\leq C\epsilon^2\,, \end{aligned} \tag{4-7}$$

where the inequality follows from (4-2), property (2) of h, and the (reverse) Schwarz inequality. Substituting (4-6) and (4-7) into (4-5) and using the mean curvature assumption we obtain

$$\Delta_\Sigma b_{p,t}(p) \le \frac{n-1}{t} + C\epsilon^2 \,.$$

Substituting this inequality into (4-4) and using property (3) of h gives

$$\Delta_\Sigma f_{\epsilon,t}(p) \le \frac{n-1}{t} + C\epsilon^2 - D\epsilon \,.$$

Now, for ϵ sufficiently small and t sufficiently large the right hand side of the above inequality is negative, and hence we arrive at the desired contradiction, $\Delta_\Sigma f_{\epsilon,t}(p) < 0$.

Proposition 4.4 has the following immediate consequences.

Corollary 4.5. *Let M be a future timelike geodesically complete spacetime which obeys the strong energy condition,* $\mathrm{Ric}(X,X) \ge 0$ *for all timelike vectors X, and let γ be a timelike S-ray. Let $W \subset I^-(\gamma) \cap I^+(S)$ be an open set on which the generalized timelike co-ray condition holds. Let Σ be a connected acausal smooth spacelike hypersurface in W with nonpositive mean curvature, $H_\Sigma \le 0$. Suppose that Σ and $\Sigma_c = \{b = c\} \cap W$ have a point in common and that $\Sigma \subset J^+(\Sigma_c, W)$. Then $\Sigma \subset \Sigma_c$.*

Corollary 4.6. *Let M be a future timelike geodesically complete spacetime which obeys the strong energy condition,* $\mathrm{Ric}(X,X) \ge 0$ *for all timelike vectors X, and let γ be a timelike S-ray. Let $W \subset I^-(\gamma) \cap I^+(S)$ be an open set on which the generalized timelike co-ray condition holds. Let Σ be a smooth spacelike hypersurface with nonpositive mean curvature, whose closure is contained in W. Assume that Σ is acausal in W and $\overline{\Sigma}$ is compact. If* $\mathrm{edge}(\Sigma) \subseteq \{b \ge c\}$*, then $\Sigma \subseteq \{b \ge c\}$.*

These corollaries, as well as the inequality (4-6), indicate that the sets $\{b \ge c\}$ are in some sense mean convex. This has been formalized in the Riemannian case by Eschenburg [E3].

5. Regions of regularity and applications

In this section we present several spacetime settings in which the generalized timelike co-ray condition is guaranteed to hold. In particular, our regularity results apply to, and provide a simplified proof of, the Lorentzian splitting theorem. We are also able to use our regularity results to improve previous results of Bartnik [B2] and Eschenburg and Galloway [EG] concerning the splitting conjecture for spatially closed spacetimes.

The co-ray condition near the base ray. Let M be a future timelike geodesically complete spacetime, and let $\gamma : [0,\infty) \to M$ be a timelike ray at $p \in M$. We prove that the generalized timelike co-ray condition always holds on an open set containing $\gamma \setminus \{p\}$. (Recall that, since γ is an S-ray with $S = \{\gamma(0)\}$, the co-ray construction makes sense.) This is an immediate consequence of the following result.

Propostion 5.1. *Let M be a future timelike geodesically complete spacetime and let $\gamma : [0,\infty) \to M$ be a future directed timelike ray. Then any generalized co-ray starting at $p = \gamma(a)$, $a > 0$, must coincide with γ.*

Proof. The basic idea here is that if the co-ray were not to coincide with γ then, by essentially a "cutting the corner argument", the maximality of γ would be violated. The proof makes use of limit curve techniques as developed in Section 2. Hence, for the proof, we will parameterize all curves with respect to the complete Riemannian metric h.

Let $\alpha : [0,\infty) \to M$ be a generalized co-ray to γ at $p = \gamma(a)$, $a > 0$. Then there is a limit maximizing sequence of future directed causal curves $\alpha_n : [0, a_n] \to M$ from p_n to $\gamma(r_n)$, with $p_n \to p$, $r_n \to \infty$, and $a_n \to \infty$, which converges to α. Let q be a point on γ to the past of p, and let q_n be a sequence of points on γ between q and p such that $q_n \to p$ and $q_n \ll p_n$ for all n. For each n sufficiently large, we can connect q_n to p_n by a timelike geodesic segment whose length approaches 0 as $n \to \infty$. We then define a new sequence $\sigma_n : [0, s_n] \to M$ by following γ from q to q_n, then going along the geodesic segment from q_n to p_n, and then following along α_n from p_n to $\gamma(r_n)$. We claim that the sequence $\{\sigma_n\}$ is limit maximizing.

To prove the claim, we let $q_n = \sigma_n(a_n)$ and $p_n = \sigma_n(b_n)$. We also fix $r > 0$ such that $p_n \in I^-(\gamma(r))$ for all n sufficiently large. Since γ is maximal and the α_n's are limit maximizing we have

$$\begin{aligned} L(\sigma_n|_{[0,s_n]}) &= L(\sigma_n|_{[0,a_n]}) + L(\sigma_n|_{[a_n,b_n]}) + L(\sigma_n|_{[b_n,s_n]}) \\ &\geq d(q, q_n) + L(\sigma_n|_{[a_n,b_n]}) + d(p_n, \gamma(r_n)) - \epsilon_n\,, \end{aligned}$$

where $\epsilon_n \to 0$. Furthermore, using the lower semicontinuity of d, we have

$$\begin{aligned} d(p_n, \gamma(r_n)) &\geq d(p_n, \gamma(r)) + d(\gamma(r), \gamma(r_n)) \\ &= d(p_n, \gamma(r)) + d(p, \gamma(r_n)) - d(p, \gamma(r)) \\ &\geq d(p, \gamma(r)) - \delta_n + d(p, \gamma(r_n)) - d(p, \gamma(r)) \\ &= -\delta_n + d(p, \gamma(r_n))\,, \end{aligned}$$

where $\delta_n \to 0$.

Combining these inequalities we obtain

$$\begin{aligned} L(\sigma_n|_{[0,s_n]}) &\geq d(q,p) - d(q_n,p) + L(\sigma_n|_{[a_n,b_n]}) + d(p, \gamma(r_n)) - \delta_n - \epsilon_n \\ &= d(q, \gamma(r_n)) - d(q_n, p) + L(\sigma_n|_{[a_n,b_n]}) - \delta_n - \epsilon_n\,, \end{aligned}$$

where $\delta_n, \epsilon_n, d(q_n, p), L(\sigma|_{(q_n,p_n)})$ all approach 0 as $n \to \infty$. This implies that the σ_n's form a limit maximizing sequence. Hence, a subsequence converges to a generalized co-ray σ. By construction, σ must consist of the portion of γ from q to p followed by α. Since σ is unbroken, α is contained in σ.

Corollary 5.2. *Let M be a future timelike geodesically complete spacetime and let $\gamma : [0,\infty) \to M$ be a future directed timelike ray at $p \in M$. Then the generalized timelike co-ray condition holds on an open set containing $\gamma \setminus \{p\}$.*

Proof. This follows from Proposition 5.1 and the fact that the generalized timelike co-ray condition is an open condition (cf. the comment after Lemma 3.3).

Thus, all the regularity results established in the preceding sections hold on a neighborhood of each point of $\gamma \setminus \{p\}$.

The relevant regularity results presented in this paper, together with the arguments of Section 3 in [G4] (with simplifications owing to completeness) or Section 5 in [N], provide a simplified proof of the Lorentzian splitting theorem. (See [G6] for a general discussion of this result and its relevance to general relativity. Here, we are focusing on Yau's version of the splitting theorem, i.e. the version requiring timelike geodesic completeness.) For the sake of completeness, and convenience to the reader, we end this subsection with the statement and a sketch of the proof of the Lorentzian splitting theorem.

Theorem. *Let M be a connected timelike geodesically complete spacetime which satisfies the strong energy condition,* $\mathrm{Ric}(X, X) \geq 0$ *for all timelike vectors X. If (M, g) contains a timelike line, then M is isometric to* $(\mathbb{R} \times S, -dt^2 \oplus h)$ *where (S, h) is a complete Riemannian manifold.*

Proof. We suppose that γ is a future directed unit speed timelike line, and let $-\gamma$ be γ with opposite orientation. We then define

$$b_r^+(x) = r - d(x, \gamma(r)), \quad b_r^-(x) = r - d(-\gamma(r), x),$$

and hence define b^+ and b^- by letting $r \to \infty$. Since $\gamma|_{[a,\infty)}$ and $-\gamma|_{[a,\infty)}$ are timelike rays for all $a \in \mathbb{R}$, it is clear from the discussion in Section 2 that b^+ and b^- are defined and finite valued on $I(\gamma) = I^+(\gamma) \cap I^-(\gamma)$. Moreover all the regularity results we've considered (and their time dual) hold on a neighborhood of each point of γ.

Now, the triangle inequality implies that $b^+ + b^- \geq 0$ on $I(\gamma)$ with equality holding along γ. Let U be a neighborhood of $\gamma(0)$ such that all the preceding regularity properties hold for $b^\pm$ on U. Look at the sets $S^\pm = \{b^\pm = 0\} \cap U$. By Proposition 4.2, $S^\pm$ are partial Cauchy surfaces in U. $S^\pm$ both pass through $\gamma(0)$, with S^- lying to the future of S^+

Let W be a small coordinate ball in S^+ centered at $\gamma(0)$ whose closure also is in S^+. By applying the fundamental existence result of Bartnik ([B1], Theorem 4.1), we get a smooth maximal spacelike hypersurface $\Sigma \subset U$ such that Σ is acausal in U with compact closure, $\mathrm{edge}(\Sigma) = \mathrm{edge}(W)$ and Σ meets γ.

By applying Corollary 4.6 and its time dual to b^+ and b^-, we have that

$$\Sigma \subseteq \{b^+ \geq 0\} \cap \{b^- \geq 0\}.$$

Corollary 4.5 now implies,

$$b^+ = b^- = 0 \text{ along } \Sigma. \tag{5-1}$$

From each point in $x \in \Sigma$ there exist timelike asymptotes $\alpha_x^\pm : [0, \infty) \to M$ to $\gamma^\pm$. Let $\alpha_x : (-\infty, \infty) \to M$ be the (possibly) broken geodesic defined by

$$\alpha_x(t) = \begin{cases} \alpha_x^-(-t), & t \leq 0, \\ \alpha_x^+(t), & t \geq 0. \end{cases}$$

Using the key equation (5-1) it follows easily that $b^+(\alpha_x(t)) = t$ and $b^-(\alpha_x(t)) = -t$, which in turn implies that α_x is a line. A similarly simple argument shows that

$\alpha_x^\pm$ are Σ-rays. This implies that $\alpha_x^\pm$ are focal point free and that α_x meets Σ orthogonally. It can further be shown that these normal geodesics do not intersect.

Now, consider the normal exponential map $E : \mathbb{R} \times \Sigma \to M$ defined by $E(t, q) = \exp tN_q$, where N is the future directed unit normal field along Σ. The discussion above implies that E is injective and nonsingular, and hence that it is a diffeomorphism onto its image.

The mean curvature function $H = H_t$ of the foliation $\Sigma_t = E(\{t\} \times \Sigma)$ obeys the following evolution equation (essentially the Raychaudhuri equation),

$$\frac{\partial H}{\partial t} = -\operatorname{Ric}(N, N) - |\nabla N|^2 \leq -\frac{1}{n-1} H^2 \,,$$

where N has been extended to be the future directed unit normal field to the Σ_t's. Now, unless ∇N vanishes identically, H_t must blow up in finite time along some normal geodesic, violating the fact that all the normal geodesics are focal point free. Thus N is parallel, and hence E is an isometry. Hence, M splits in the required way in a neighborhood of γ. This splitting can then be continued to all of M in a fairly straightforward manner (see, e.g. [E2]).

The co-ray condition and spacelike hypersurfaces. Let M be a future timelike geodesically complete spacetime, and suppose S is an acausal spacelike hypersurface in M which admits an S-ray γ. (The existence of an S-ray is guaranteed if S is compact.) It is easily verified in this situation that $b = b_\gamma$ and d are finite valued on the open set $I^-(\gamma) \cap I^+(D^-(S)) = I^-(\gamma) \cap [J^+(S) \cup D^-(S)]$, where $D^-(S)$ is the past domain of dependence of S. For instance, if $x \in D^-(S)$ and $y \in J^+(S)$ then

$$d(x, y) \leq d(x, S) + d(S, y) < \infty \,. \tag{5-2}$$

The finiteness of $d(x, S) = \sup_{z \in S} d(x, z)$ follows from basic properties of the domain of dependence (which, for instance, insure that d is continuous on $D(S)$). Furthermore, by setting $y = \gamma(r)$ in (5-2) one easily obtains,

$$b(x) \geq -d(x, S) \text{ for all } x \in I^-(\gamma) \cap D^-(S) \,.$$

The finiteness of b and d on $I^-(\gamma) \cap [J^+(S) \cup D^-(S)]$ and the fact that any future inextendible causal curve starting in $D^-(S)$ must enter $I^+(S)$ are all that is needed to extend the previous regularity results to the region $I^-(\gamma) \cap [J^+(S) \cup D^-(S)]$.

Proposition 5.3. *Let M be a future timelike geodesically complete spacetime, and suppose γ is an S-ray, where S is an acausal spacelike hypersurface in M. Then all the previous regularity results which refer to the region $I^-(\gamma) \cap I^+(S)$ are in fact valid on the larger region $I^-(\gamma) \cap [J^+(S) \cup D^-(S)]$.*

In what follows we will be particularly interested in the case in which S is compact. In this case it was shown in [EG] (Lemma 5) that the generalized timelike co-ray condition holds on $I^-(\gamma) \cap J^+(S)$. However, the observations made above are applicable here, as well, i.e. essentially the same argument as that given in [EG] shows that the generalized timelike co-ray condition holds on $I^-(\gamma) \cap [J^+(S) \cup D^-(S)]$. Hence, we have the following.

Lemma 5.4. *Let M be a future timelike geodesically complete spacetime which contains a compact acausal spacelike hypersurface S, and let γ be an S-ray. Then the generalized timelike co-ray condition holds on $I^-(\gamma) \cap [J^+(S) \cup D^-(S)]$.*

In particular, $b = b_\gamma$ is continuous on $I^-(\gamma) \cap [J^+(S) \cup D^-(S)]$. We now show that b extends continuously to the boundary of $I^-(\gamma)$ (i.e., b tends to infinity as one approaches $\partial I^-(\gamma)$).

Proposition 5.5. *Let M be a future timelike geodesically complete spacetime which contains a compact acausal spacelike hypersurface S, and let γ be an S-ray. Then $b : J^+(S) \cup D^-(S) \to (-\infty, \infty]$ is continuous. Moreover, the level sets $\Sigma_c = \{b = c\} \cap [J^+(S) \cup D^-(S)]$ are partial Cauchy surfaces in $J^+(S) \cup D^-(S)$.*

Proof. Since $b = \infty$ on $M \setminus I^-(\gamma)$, to establish the claim concerning the continuity of b, it is sufficient to show the following: if z_n is a sequence in $I^-(\gamma) \cap [J^+(S) \cup D^-(S)]$ and $z_n \to z \in \partial I^-(\gamma) \cap [J^+(S) \cup D^-(S)]$, then $b(z_n) \to \infty$. Suppose to the contrary that there is a sequence $z_n \to z \in \partial I^-(\gamma)$ such that $b(z_n) \le c < \infty$. Then one can find sequences $r_n \to \infty$ and $\delta_n \to 0$ such that $z_n \in I^-(\gamma(r_n))$ and

$$b_n(z_n) \le c + \delta_n \quad \text{for all } n\,, \tag{5-3}$$

where $b_n = b_{r_n} = d(\gamma(0), \gamma(r_n)) - d(., \gamma(r_n))$.

Let $\alpha_n : [0, a_n] \to M$ be a limit maximizing sequence of causal curves from z_n to $\gamma(r_n)$. By applying Lemmas 2.1 and 2.4, and passing to a subsequence if necessary, we may assume that $\{\alpha_n\}$ converges to a causal ray $\alpha : [0, \infty) \to M$, with $\alpha(0) = z$. Since $\alpha \subset \overline{I^-(\gamma)}$ and $\alpha(0) \in \partial I^-(\gamma)$, it follows that $\alpha \subset \partial I^-(\gamma)$. In fact, by properties of achronal boundaries (cf., [HE] or [P]) α must be a null geodesic generator of $\partial I^-(\gamma)$. We prove the following claim.

Claim. For all $x \in I^-(\alpha) \cap [J^+(S) \cup D^-(S)]$, $b(x) \le c$.

To prove the claim, choose $s > 0$ so that $x \in I^-(\alpha(s))$. Then, for n sufficiently large, we have $x \in I^-(\alpha_n(s))$ and, by (2-5),

$$b_n(x) \le b_n(\alpha_n(s))\,. \tag{5-4}$$

Since $\{\alpha_n\}$ is limit maximizing, we obtain

$$\begin{aligned} b_n(\alpha_n(s)) - b_n(z_n) &= d(z_n, \gamma(r_n)) - d(\alpha_n(s), \gamma(r_n)) \\ &\le d(z_n, \alpha_n(s)) + \epsilon_n\,, \end{aligned} \tag{5-5}$$

where $\epsilon_n \to 0$. Inequalities (5-3), (5-4) and (5-5) combine to give

$$b_n(x) \le c + d(z_n, \alpha_n(s)) + \delta_n + \epsilon_n\,.$$

By letting $n \to \infty$ and applying (2-2), the claim is established.

The proof now proceeds as in the proof of Lemma 5 in [EG]. Choose a sequence $t_n \to \infty$ and set $p_n = \alpha(t_n)$; we may assume $p_n \in J^+(S)$ for all n. By using the completeness assumption and the compactness of S it can be shown that $d(S, p_n) \to \infty$. By perturbing the sequence $\{p_n\}$ slightly to the past and using the lower semicontinuity of d, one easily constructs a sequence $\{q_n\} \subset I^-(\gamma) \cap J^+(S)$ such that $q_n \in I^-(p_n)$ for all n and $d(S, q_n) \to \infty$. The inequality (2-5) then implies that $b(q_n) \to \infty$, which is a contradiction, since, by the claim, $b(q_n) \leq c$ for all n.

Now consider the level set $\Sigma_c = \{b = c\} \cap [J^+(S) \cup D^-(S)]$. Proposition 4.2 and the comment that follows, together with Proposition 5.3 and Lemma 5.4 imply that Σ_c is an acausal spacelike hypersurface in M. By the continuity of b just established, Σ_c is closed, and hence edgeless, in $J^+(S) \cup D^-(S)$. Thus, Σ_c is a partial Cauchy surface in $J^+(S) \cup D^-(S)$.

In the globally hyperbolic case we obtain global regularity.

Corollary 5.6. *Let M be a globally hyperbolic, future timelike geodesically complete spacetime which contains a compact spacelike hypersurface S, and let γ be an S-ray. Then $b : M \to (-\infty, \infty]$ is continuous. Moreover, the level sets $\Sigma_c = \{b = c\}$ are partial Cauchy surfaces in M.*

Proof. Since M is globally hyperbolic and S is compact, S must be Cauchy (cf., [Bu+], [G2]), in which case $J^+(S) \cup D^-(S) = M$.

Comment. De Sitter space is a good example of a globally hyperbolic, geodesically complete spacetime with S-rays γ (where S is compact and Cauchy) for which $I^-(\gamma) \neq M$. Thus, in general, the Busemann function as considered in Corollary 5.6 will not be globally finite valued.

The splitting conjecture for spatially closed spacetimes. We consider the following conjecture of R. Bartnik [B2].

Conjecture. *Let M be a globally hyperbolic spacetime which contains a compact spacelike hypersurface S and obeys the strong energy condition, $Ric(X, X) \geq 0$ for all timelike vectors X. If M is timelike geodesically complete, then M splits isometrically into the product $(R \times V, -dt^2 \oplus h)$, where (V, h) is a compact Riemannian manifold.*

The conjecture should be interpreted as a statement about the rigidity of the Hawking-Penrose singularity theorems: unless spacetime splits (and hence is static), spacetime must be singular, i.e., timelike geodesically incomplete. The idea that spatially closed spacetimes should fail to be singular only under exceptional circumstances was discussed by Geroch [GR2] long ago. Following an approach of Geroch [GR1], Bartnik proved the conjecture under the additional assumption that there exists a point p such that $M \setminus [I^+(p) \cup I^-(p)]$ is compact. Bartnik's result improved an earlier result of Galloway [G1]. More recently, Eschenburg and Galloway [EG] proved the conjecture under the additional assumption that there exists an S-ray γ whose past contains S, $I^-(\gamma) \supset S$. Moreover, for this result, the assumption of global hyperbolicity is not needed; one merely needs to assume that S is acausal.

Using the regularity theory developed in this paper we are able to improve the aforementioned results as follows.

Theorem 5.7. *Let M be a spacetime which contains a compact acausal spacelike hypersurface S and obeys the strong energy condition. If M is timelike geodesically complete and contains a future S-ray γ and a past S-ray η such that $I^-(\gamma) \cap I^+(\eta) \neq \emptyset$ then M splits as in the conjecture.*

Every compact acausal spacelike hypersurface S admits a past and future S-ray. When M is globally hyperbolic, Bartnik's condition, $M \setminus [I^+(p) \cup I^-(p)]$ is compact for some p, is easily seen to imply the condition on the S-rays in Theorem 5.7.

We briefly comment on the proof of Theorem 5.7. Details will appear in a forthcoming paper. The level sets $\Sigma_+ = \{b_\gamma = a\} \cap D(S)$ and $\Sigma_- = \{b_\eta =$

$c\} \cap D(S)$ are partial Cauchy surfaces in $D(S)$, the domain of dependence of S. Using the condition on the S-rays, along with other assumptions, it is shown that, for appropriate choices of a and c, $J^-(\Sigma_+) \cap J^+(\Sigma_-)$ is compact with nonempty interior (so that $d(\Sigma_-, \Sigma_+) > 0$). Now, by using the fact that Σ_+ is mean convex to the future and Σ_- is mean convex to the past (in the sense of support functions, of course) it can be shown that Σ_- and Σ_+ are compact and totally geodesic, and that $J^-(\Sigma_+) \cap J^+(\Sigma_-)$ is isometric to the Lorentzian product $[0, \ell] \times \Sigma_-$. This latter rigidity result, which is perhaps of independent interest, generalizes a theorem of Claus Gerhardt ([GC], Theorem 7.4). This is enough to prove Theorem 5.7, since the conjecture is well known to hold if M contains a compact maximal spacelike hypersurface (cf. [B2], [G5]).

Final remarks.

We mention one additional situation in which the generalized timelike co-ray condition can be shown to hold. Let M be a future timelike geodesically complete spacetime, and let γ be a future directed timelike ray starting at $p \in M$. Let us say that condition (S) holds on $A \subset I^+(p)$ provided for each future directed null ray $\eta : [0, \infty) \to M$ starting in A, $d(p, \eta(t)) \to \infty$ as $t \to \infty$. For example, if M is Minkowski space then condition (S) holds on $I^+(p)$ for all p. In [S], Seifert argues that condition (S) will hold on $I^+(p)$ provided M is both future timelike and future null geodesically complete (cf., Lemma 9 in [S]; although no causality condition is assumed, clearly some such condition is needed). If condition (S) is satisfied on $I^-(\gamma) \cap I^+(p)$ then one can use Lemma 2 in [EG] as in the proof of Lemma 5 in [EG] to show that the generalized timelike co-ray condition holds on $I^-(\gamma) \cap I^+(p)$, thereby establishing the regularity of b_γ on this region.

In our treatment of the regularity of Lorentzian Busemann functions we have considered spacetimes which are future timelike geodesically complete. As mentioned in the introduction, a similar treatment can be carried out for spacetimes which are assumed to be globally hyperbolic, but not necessarily future timelike geodesically complete. In this case the treatment of a number of topics (such as the existence of maximizers and the behavior of the timelike cut locus) can be greatly simplified or altogether eliminated. However the proofs of the results which make use of support functions (e.g., Proposition 4.4) must be modified because the support functions $b_{p,t}$, where t is Lorentzian arc length along the asymptote, may no longer be defined for large t. As long as the given base ray γ is future complete one can proceed along the lines of [G4] by approximating b by the pre-Busemann functions b_r and introducing suitable support functions for the b_r's. Even if γ is not future complete this method can be useful (cf. [AH]). Finally, whether or not spacetime is assumed to be globally hyperbolic, certain results (e.g., Lemma 5.4, Proposition 5.5, Corollary 5.6) require future completeness.

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Department of Mathematics, University of Miami, Coral Gables, Florida 33124
E-mail address: galloway@math.miami.edu

National Security Agency, Fort Meade, Maryland 20755-6000
E-mail address: ahorta@ix.netcom.com